

Fluctuation-dissipation theorem and quantum tunneling with dissipation at finite temperature

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A reformulation of the fluctuation-dissipation theorem of Callen and Welton [Phys. Rev. **83**, 34 (1951)] is presented in such a manner that the basic idea of Feynman and Vernon [Ann. Phys. (N.Y.) **24**, 118 (1963)] and Caldeira and Leggett [Ann. Phys. (N.Y.) **149**, 374 (1983); Phys. Rev. Lett. **46**, 211 (1981)] of using an infinite number of oscillators to simulate the dissipative medium is realized manifestly without actually introducing oscillators. If one assumes the existence of a well defined dissipative coefficient $R(\omega)$, which depends little on the temperature in the energy region we are interested in, the spontaneous and induced emissions as well as induced absorption of these effective oscillators with a correct Bose distribution automatically appear. Combined with a dispersion relation, we reproduce the tunneling formula in the presence of dissipation at a finite temperature without referring to an explicit model Lagrangian. The fluctuation-dissipation theorem of Callen and Welton is also generalized to the fermionic dissipation (or fluctuation), which allows a transparent physical interpretation in terms of second-quantized fermionic oscillators. This fermionic version of the fluctuation-dissipation theorem may become relevant in the analyses of, for example, fermion radiation from a black hole and also supersymmetry in the early Universe. [S1063-651X(98)04812-0]

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I. INTRODUCTION

The fluctuation-dissipation theorem, which relates the spontaneous fluctuation of “force fields” in thermal equilibrium to irreversible dissipation, provides a basis of statistical mechanics for irreversible processes that are slightly out of thermal equilibrium. The fluctuation-dissipation theorem has been formulated by various authors in the past [1–7]. We find the formulation by Callen and Welton [2] to be intuitively understandable and appealing. They showed [2] that a general form of the fluctuation-dissipation theorem covers a wide range of phenomena such as the Einstein relation for Brownian motion [8], the Nyquist formula for voltage fluctuation in conductors [9], and the Planck distribution for photons. In Ref. [10] it was shown that the effect of dissipation on quantum tunneling (or coherence) at *zero temperature* can be formulated on the basis of the fluctuation-dissipation theorem of Callen and Welton, causality, and unitarity (i.e., dispersion relations) without referring to an explicit form of Lagrangian in the manner of Caldeira and Leggett [11].

In this paper we present a reformulation of the fluctuation-dissipation theorem of Callen and Welton in such a manner that the basic idea of Feynman and Vernon [7] and Caldeira and Leggett [11], which simulates the dissipative medium by an infinite number of oscillators, becomes manifest without actually introducing oscillators. The quantum tunneling at *finite temperature* is described by this reformulation. Although we use the quantum mechanical Fermi golden rule, the spontaneous and induced emissions as well as induced absorption of these effective oscillators with a correct Bose distribution automatically appear if one assumes the existence of a well defined dissipative coefficient $R(\omega)$, which depends little on the temperature in the energy region we are interested in.

To be specific, for the simplest case of a Hermitian (composite) operator Q , which represents the (generally complicated) dynamical freedom of the dissipative medium, we have the relations

$$\begin{aligned} & \frac{2}{\pi} \frac{\hbar\omega}{2} \left[1 + \frac{1}{e^{\beta\hbar\omega} - 1} \right] R(\omega) \\ &= \hbar \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar\omega) \\ & \quad \times |\langle E + \hbar\omega | Q | E \rangle|^2 \end{aligned} \quad (1)$$

and

$$\begin{aligned} & \frac{2}{\pi} \frac{\hbar\omega}{2} \left[\frac{1}{e^{\beta\hbar\omega} - 1} \right] R(\omega) \\ &= \hbar \int_0^\infty dE \rho(E) f(E) \rho(E - \hbar\omega) \\ & \quad \times |\langle E - \hbar\omega | Q | E \rangle|^2, \end{aligned} \quad (2)$$

where $f(E)$ stands for a normalized Boltzmann factor with $f(E + \hbar\omega)/f(E) = e^{-\beta\hbar\omega}$ and $\beta = 1/kT$. These formulas are remarkable. On the right-hand sides of these relations we simply use Fermi’s golden rule for the first-order perturbation. We recognize the left-hand side of Eq. (1) as representing the spontaneous and induced emissions of second-quantized bosonic oscillators and the left-hand side of Eq. (2) as the standard (induced) absorption formula. Moreover, the spectrum of the effective bosonic oscillators is characterized by the dissipative coefficient (resistance) $R(\omega)$. In other words, the presence of $R(\omega)$ necessarily implies the presence of effective oscillators characterized by $R(\omega)$, as was emphasized in Ref. [10]. These expressions realize the basic

idea of Feynman and Vernon [7] and Caldeira and Leggett [11] without actually introducing an infinite number of oscillators and they naturally satisfy the detailed balancing relation. It is shown that these formulas (1) and (2) lead to the conventional fluctuation-dissipation theorem of Callen and Welton.

Combining Eqs. (1) and (2) with a dispersion relation, it is shown in the text that we can reproduce the tunneling formula in the presence of dissipation at finite temperature without referring to an explicit model Lagrangian. We also present a generalization of the Callen-Welton formula for fermionic dissipation (or fluctuation) in contrast to the conventional bosonic dissipation such as in Eqs. (1) and (2). These formulas for fermionic dissipation, though somewhat academic at this moment, may become relevant in the analyses of, for example, fermion emission from a black hole or supersymmetric properties in the early Universe.

II. REFORMULATION OF THE FLUCTUATION-DISSIPATION THEOREM

A. Microscopic power dissipation

We first start with a Hamiltonian

$$H = H_0(Q) + V(Q, q)e^{i\omega t} + V(Q, q)^\dagger e^{-i\omega t}, \quad (3)$$

where $H_0(Q)$ is the unperturbed Hamiltonian for the dissipative medium, which has eigenstates

$$H_0|E_n\rangle = E_n|E_n\rangle. \quad (4)$$

The variable q appearing in $V(Q, q)$ describes the external dynamical freedom that perturbs the dissipative medium. The variable q is treated as a classical variable for the moment.

If the dissipative medium is initially in the state $|E_n\rangle$, the lowest-order transition rate (transition probability per unit time) is given by Fermi's golden rule by treating the last two terms in Eq. (3) as a small perturbation,

$$w = \frac{2\pi}{\hbar} [|\langle E_n + \hbar\omega | V^\dagger | E_n \rangle|^2 \rho(E_n + \hbar\omega) + |\langle E_n - \hbar\omega | V | E_n \rangle|^2 \rho(E_n - \hbar\omega)]. \quad (5)$$

Since the first term stands for the absorption of energy $\hbar\omega$ and the second term for the emission of energy $\hbar\omega$, the energy absorption rate by the dissipative medium is given by

$$2\pi\omega [|\langle E_n + \hbar\omega | V^\dagger | E_n \rangle|^2 \rho(E_n + \hbar\omega) - |\langle E_n - \hbar\omega | V | E_n \rangle|^2 \rho(E_n - \hbar\omega)]. \quad (6)$$

If the system is initially in thermal equilibrium at temperature T , we must average over all initial states, weighting the state $|E_n\rangle$ by the (normalized) Boltzmann factor $f(E_n)$, which satisfies

$$\frac{f(E_n + \hbar\omega)}{f(E_n)} = e^{-\beta\hbar\omega}, \quad \beta = \frac{1}{kT}. \quad (7)$$

Then the energy dissipation per unit time is given by

$$P(\omega) = 2\pi\omega \int_0^\infty dE \rho(E) f(E) [|\langle E + \hbar\omega | V^\dagger | E \rangle|^2 \rho(E + \hbar\omega) - |\langle E - \hbar\omega | V | E \rangle|^2 \rho(E - \hbar\omega)], \quad (8)$$

where we replaced the summation over n by an integration over energy.

Now we assume that V can be written as $V = qQ/2$, namely, the interaction part in Eq. (3) is written as

$$H_I = \frac{1}{2}(qe^{i\omega t}Q + Q^\dagger \bar{q}e^{-i\omega t}) \quad (9)$$

in the spirit of linear response approximation; q is an infinitesimal complex number and Q is a bosonic composite operator, respectively. Here we allow the operator Q to be non-Hermitian in general so that we can readily extend our formulation to fermionic dissipation later. Then we obtain

$$P(\omega) = \frac{\pi\omega}{2} q\bar{q} \int_0^\infty dE \rho(E) f(E) [|\langle E + \hbar\omega | Q^\dagger | E \rangle|^2 \times \rho(E + \hbar\omega) - |\langle E - \hbar\omega | Q | E \rangle|^2 \rho(E - \hbar\omega)]. \quad (10)$$

B. Macroscopic dissipative coefficient $R(\omega)$

We next define the phenomenological macroscopic dissipative coefficient (resistance) $R(\omega)$ on the basis of the following reasoning [10]. We first define an infinitesimal (complex) coordinate in H_I [Eq. (9)],

$$q(t) = qe^{i\omega t}. \quad (11)$$

The existence of the energy dissipation into the medium (10) induced by the external perturbation $q(t)$ suggests the presence of a dissipative force (reaction) acting on the variable $\text{Re } q(t)$, which oscillates with frequency ω ,

$$F = -R(\omega)\text{Re } \dot{q}(t), \quad (12)$$

where $R(\omega)$ is a real function. Note that $H_I = Q\text{Re } q(t)$ in Eq. (9) for a Hermitian operator Q ; this shows that $\text{Re } q(t)$ is a natural classical counter part of the (Hermitian) quantum variable \hat{q} to describe the macroscopic quantum system $H_0(q)$. The power dissipation per unit time generated by this phenomenological reactive force is given by

$$\begin{aligned} P(\omega) &= -\overline{\text{Re } F \text{Re } \dot{q}(t)} \\ &= R(\omega) \overline{[\text{Re } \dot{q}(t)]^2} \\ &= \frac{1}{2} R(\omega) \overline{[\text{Re } \dot{q}(t)]^2 + [\text{Im } \dot{q}(t)]^2} \\ &= \frac{\omega^2}{2} R(\omega) q\bar{q}, \end{aligned} \quad (13)$$

where the overbar indicates time averaging.

Combining Eqs. (10) and (13), we obtain the microscopic expression for the dissipative coefficient (resistance) $R(\omega)$,

$$\begin{aligned}
R(\omega) &= \frac{\pi}{\omega} \int_0^\infty dE \rho(E) f(E) [|\langle E + \hbar\omega | Q^\dagger | E \rangle|^2 \\
&\quad \times \rho(E + \hbar\omega) - |\langle E - \hbar\omega | Q | E \rangle|^2 \rho(E - \hbar\omega)] \\
&= \frac{\pi}{\omega} (1 - e^{-\beta\hbar\omega}) \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar\omega) \\
&\quad \times |\langle E + \hbar\omega | Q^\dagger | E \rangle|^2. \tag{14}
\end{aligned}$$

From this expression of $R(\omega)$ we find the basic relations

$$\frac{2}{\pi} \frac{\hbar\omega}{2} \left[1 + \frac{1}{e^{\beta\hbar\omega} - 1} \right] R(\omega) = \hbar \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar\omega) \times |\langle E + \hbar\omega | Q^\dagger | E \rangle|^2, \tag{15}$$

$$\frac{2}{\pi} \frac{\hbar\omega}{2} \left[\frac{1}{e^{\beta\hbar\omega} - 1} \right] R(\omega) = \hbar \int_0^\infty dE \rho(E) f(E) \rho(E - \hbar\omega) \times |\langle E - \hbar\omega | Q | E \rangle|^2. \tag{16}$$

Equation (15) is the absorption of $\hbar\omega$ by Q^\dagger and Eq. (16) the emission of $\hbar\omega$ by Q .

These relations (15) and (16) are the prototype of the fluctuation-dissipation theorem of Callen and Welton. The fluctuation-dissipation theorem as it stands is a mathematical identity and contains no physical contents by itself. What is remarkable is that we obtain highly nontrivial relations in Eqs. (15) and (16) if one *assumes* that the dissipative coefficient $R(\omega)$ depends little on the temperature in the region we are interested in. We can recognize the left-hand side of Eq. (15) as standing for the spontaneous and induced emissions of the second-quantized bosonic oscillator with frequency ω into the dissipative medium, whereas the left-hand side of Eq. (16) is recognized as the (induced) absorption of these oscillators from the dissipative medium at temperature T . Moreover, the spectrum of these effective oscillators is precisely specified by the dissipative coefficient $R(\omega)$. In particular, there is no effective oscillator of ω for which $R(\omega)$ vanishes. These properties realize the basic physical idea of Feynman and Vernon [7] and Caldeira and Leggett [11], which simulates the dissipative medium by an infinite number of oscillators. In our approach this physical idea is realized by a simple application of Fermi's golden rule combined with the temperature independence of the dissipative coefficient $R(\omega)$ without actually introducing oscillators. We emphasize that these oscillators are effective and such real oscillators do not exist inside the dissipative medium in general. It is shown in Sec. II C that the quantum tunneling with dissipation at finite temperature is formulated on the basis of Eqs. (15) and (16) and dispersion relations (i.e., unitarity and causality) without referring to the explicit model Lagrangian of Caldeira and Leggett.

It is obvious from Eq. (3) that the change $\omega \rightarrow -\omega$ corresponds to the exchange $V \leftrightarrow V^\dagger$. By the definition of $R(\omega)$ in Eq. (46), we then obtain the expression

$$\begin{aligned}
R(-\omega) &= \frac{\pi}{\omega} \int_0^\infty dE \rho(E) f(E) [|\langle E + \hbar\omega | Q | E \rangle|^2 \\
&\quad \times \rho(E + \hbar\omega) - |\langle E - \hbar\omega | Q^\dagger | E \rangle|^2 \rho(E - \hbar\omega)] \tag{17}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{\omega} (1 - e^{-\beta\hbar\omega}) \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar\omega) \\
&\quad \times |\langle E + \hbar\omega | Q | E \rangle|^2. \tag{18}
\end{aligned}$$

Clearly, $R(-\omega) = R(\omega)$ in the case of a Hermitian operator, $Q = Q^\dagger$.

For $R(-\omega)$ with $\omega > 0$, we obtain relations similar to Eqs. (15) and (16),

$$\begin{aligned}
\frac{2}{\pi} \frac{\hbar\omega}{2} \left[1 + \frac{1}{e^{\beta\hbar\omega} - 1} \right] R(-\omega) \\
= \hbar \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar\omega) |\langle E + \hbar\omega | Q | E \rangle|^2, \tag{19}
\end{aligned}$$

$$\begin{aligned}
\frac{2}{\pi} \frac{\hbar\omega}{2} \left[\frac{1}{e^{\beta\hbar\omega} - 1} \right] R(-\omega) \\
= \hbar \int_0^\infty dE \rho(E) f(E) \rho(E - \hbar\omega) |\langle E - \hbar\omega | Q^\dagger | E \rangle|^2. \tag{20}
\end{aligned}$$

Equation (19) is the absorption of $\hbar\omega$ by Q and Eq. (20) the emission of $\hbar\omega$ by Q^\dagger from the dissipative medium, respectively.

C. Fluctuation-dissipation theorem

Finally, we formulate the fluctuation-dissipation theorem of Callen and Welton. From Eqs. (15) and (20) we have, after integration over ω (and changing the order of integration over E and ω),

$$\begin{aligned}
\langle QQ^\dagger \rangle &\equiv \int_0^\infty \langle E | QQ^\dagger | E \rangle \rho(E) f(E) dE \\
&= \int_0^\infty dE \rho(E) f(E) \left\{ \int_0^\infty |\langle E + \hbar\omega | Q^\dagger | E \rangle|^2 \right. \\
&\quad \times \rho(E + \hbar\omega) d(\hbar\omega) + \int_0^\infty |\langle E - \hbar\omega | Q^\dagger | E \rangle|^2 \\
&\quad \times \rho(E - \hbar\omega) d(\hbar\omega) \left. \right\} \\
&= \frac{2}{\pi} \int_0^\infty d\omega \frac{\hbar\omega}{2} \left\{ \left[1 + \frac{1}{e^{\beta\hbar\omega} - 1} \right] R(\omega) \right. \\
&\quad \left. + \left[\frac{1}{e^{\beta\hbar\omega} - 1} \right] R(-\omega) \right\} \tag{21}
\end{aligned}$$

and similarly from Eqs. (16) and (19),

$$\begin{aligned} \langle Q^\dagger Q \rangle = & \frac{2}{\pi} \int_0^\infty d\omega \frac{\hbar\omega}{2} \left\{ \left[\frac{1}{e^{\beta\hbar\omega} - 1} \right] R(\omega) \right. \\ & \left. + \left[1 + \frac{1}{e^{\beta\hbar\omega} - 1} \right] R(-\omega) \right\}. \end{aligned} \quad (22)$$

Furthermore,

$$\frac{1}{2} \langle Q^\dagger Q + Q Q^\dagger \rangle = \frac{2}{\pi} \int_0^\infty d\omega E(\omega, T) \left[\frac{R(\omega) + R(-\omega)}{2} \right], \quad (23)$$

where

$$E(\omega, T) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}. \quad (24)$$

If Q is a Hermitian operator, $Q = Q^\dagger$, we have $R(-\omega) = R(\omega)$ and Eqs. (21)–(23) reduce to the familiar form of the fluctuation-dissipation theorem of Callen and Welton [2],

$$\langle Q^2 \rangle = \frac{2}{\pi} \int_0^\infty d\omega E(\omega, T) R(\omega). \quad (25)$$

The fluctuation $\langle Q^2 \rangle$ on the left-hand side is expressed in terms of the dissipative coefficient $R(\omega)$ with a universal kernel $E(\omega, T)$. The fluctuation-dissipation theorem is regarded as a sum rule relating two quantities $\langle Q^2 \rangle$ and $R(\omega)$, which are measured independently.

In Ref. [10] we used the final formula (25) and thus only the *zero-temperature* case of quantum tunneling with dissipation was formulated in a model-independent manner. The relations (15) and (16) apparently contain more information and they allow us to formulate the tunneling with dissipation at finite temperature.

III. QUANTUM TUNNELING WITH DISSIPATION AT FINITE TEMPERATURE

We start with the total Hamiltonian

$$H = H_0(Q) + H_0(q) + H_I(q, Q), \quad (26)$$

where $H_0(q)$ describes the unperturbed Hamiltonian of the quantum system we are interested in,

$$H_0(q) = \frac{1}{2M} p^2 + V(q), \quad (27)$$

and $H_I(q, Q)$ stands for the interaction Hamiltonian in Eq. (10),

$$H_I(q, Q) = qQ, \quad (28)$$

but now the variable q is promoted to a Hermitian quantum operator; the explicit time dependence disappears in the Schrödinger picture. We also choose a Hermitian Q , $Q^\dagger = Q$, in conformity with the convention of the standard Caldeira-Leggett model [11]. In the context of macroscopic quantum tunneling (or to be more precise, quantum coher-

ence), we choose $V(q)$ in Eq. (28) as a symmetric double-well potential. $H_0(Q)$ describes the dissipative medium as before, but we do not need an explicit form of $H_0(Q)$ and the dynamical properties of $H_0(Q)$ are indirectly specified by our fluctuation-dissipation theorem (15) and (16).

We now start with the eigenstates of $H_0(q)$

$$H_0(q)|n\rangle = E_n|n\rangle \quad (29)$$

and consider the transition probability for $n \rightarrow m + \hbar\omega$ by emitting energy $\hbar\omega$ to the dissipative medium, which is assumed to be in thermal equilibrium with temperature T . The transition probability for this process is given by the lowest-order perturbation of $H_I(q, Q)$ as

$$\begin{aligned} w(n \rightarrow m + \hbar\omega) = & \frac{2\pi}{\hbar} | \langle m|q|n \rangle |^2 \int_0^{\hbar\Lambda} \left[\int_0^\infty \rho(E + \hbar\omega) \right. \\ & \left. \times | \langle E + \hbar\omega | Q | E \rangle |^2 \rho(E) f(E) dE \right] \\ & \times \delta(E_n - E_m - \hbar\omega) d(\hbar\omega) \\ = & \frac{2\pi}{\hbar} | \langle m|q|n \rangle |^2 \frac{2}{\pi} \int_0^{\hbar\Lambda} \frac{\hbar\omega}{2} \left[1 + \frac{1}{e^{\hbar\omega/kT} - 1} \right] \\ & \times \frac{R(\omega)}{\hbar} \delta(E_n - E_m - \hbar\omega) d(\hbar\omega), \end{aligned} \quad (30)$$

where we used Eq. (15) for a Hermitian Q . We also introduced an explicit cutoff $\hbar\Lambda$ of effective frequency of the dissipative medium, which could be included in the definition of $R(\omega)$.

At finite temperature we also have an absorption probability

$$\begin{aligned} w(n + \hbar\omega \rightarrow m) = & \frac{2\pi}{\hbar} | \langle m|q|n \rangle |^2 \int_0^{\hbar\Lambda} \left[\int_0^\infty \rho(E - \hbar\omega) \right. \\ & \left. \times | \langle E - \hbar\omega | Q | E \rangle |^2 \rho(E) f(E) dE \right] \\ & \times \delta(E_n - E_m + \hbar\omega) d(\hbar\omega) \\ = & \frac{2\pi}{\hbar} | \langle m|q|n \rangle |^2 \frac{2}{\pi} \int_0^{\hbar\Lambda} \frac{\hbar\omega}{2} \\ & \times \left[\frac{1}{e^{\hbar\omega/kT} - 1} \right] \frac{R(\omega)}{\hbar} \\ & \times \delta(E_n - E_m + \hbar\omega) d(\hbar\omega), \end{aligned} \quad (31)$$

where we used Eq. (16). It is interesting that these formulas satisfy the detailed balancing relation

$$e^{\hbar\omega/kT} w(n \rightarrow m + \hbar\omega) = w(m + \hbar\omega \rightarrow n), \quad (32)$$

with $\hbar\omega = E_n - E_m$.

We next define the half-width of the state $|n\rangle$ for emission (when we prepare the state $|n\rangle$ at $t=0$)

$$\frac{1}{2}\Gamma_n^{(+)} = \frac{1}{2}\hbar \sum_m w(n \rightarrow m + \hbar\omega) \quad (33)$$

and the corresponding one for absorption

$$\frac{1}{2}\Gamma_n^{(-)} = \frac{1}{2}\hbar \sum_m w(n + \hbar\omega \rightarrow m). \quad (34)$$

It can be confirmed that Eq. (33) gives $(1/2)\Gamma_n = (1/2)\hbar\eta/M$ at $T=0$ for a simple harmonic oscillator $H_0(q) = (1/2M)p^2 + (M\omega^2/2)q^2$ and Ohmic dissipation $R(\omega) = \eta$ [10]; this expression of Γ_n is consistent with a damped oscillator $M\ddot{q} + \eta\dot{q} + M\omega^2q = 0$, which in turn justifies the normalization of H_I in Eq. (28).

The basic idea in our attempt to reproduce the results of the Caldeira-Leggett model without introducing an auxiliary infinite number of oscillators is to write dispersion relations, which relate the imaginary part of energy eigenvalue to the corresponding real part. The imaginary parts are evaluated by means of the fluctuation-dissipation theorem as in Eqs. (33) and (34).

We thus write a generalization of the dispersion relation for the self-energy correction $\Sigma_n(E)$ to the energy eigenvalue E_n as

$$\begin{aligned} \Sigma_n(E) &= \frac{1}{\pi} \int_0^\infty \frac{\text{Im} \Sigma_n^{(+)}(E') dE'}{E' - E - i\epsilon} \\ &+ \frac{1}{\pi} \int_{-\hbar\Lambda}^\infty \frac{\text{Im} \Sigma_n^{(-)}(E') dE'}{E' - E - i\epsilon}, \end{aligned} \quad (35)$$

where

$$\begin{aligned} \text{Im} \Sigma_n^{(+)}(E) &\equiv \frac{1}{2}\Gamma_n^{(+)}(E) \\ &= \sum_m |\langle m|q|n \rangle|^2 \int_0^{\hbar\Lambda} \hbar\omega \left[1 + \frac{1}{e^{\hbar\omega/kT} - 1} \right] \frac{R(\omega)}{\hbar} \\ &\quad \times \delta(E - E_m - \hbar\omega) d(\hbar\omega) \end{aligned} \quad (36)$$

and

$$\begin{aligned} \text{Im} \Sigma_n^{(-)}(E) &\equiv \frac{1}{2}\Gamma_n^{(-)}(E) = \sum_m |\langle m|q|n \rangle|^2 \\ &\quad \times \int_0^{\hbar\Lambda} \hbar\omega \left[\frac{1}{e^{\hbar\omega/kT} - 1} \right] \frac{R(\omega)}{\hbar} \\ &\quad \times \delta(E - E_m + \hbar\omega) d(\hbar\omega). \end{aligned} \quad (37)$$

Note that the lower bound of the integration range in the second term in Eq. (35) starts at $-\hbar\Lambda$ due to the definition in Eq. (37). We thus obtain

$$\begin{aligned} \Sigma_n(E) &= \sum_m |\langle m|q|n \rangle|^2 \frac{1}{\pi} \int_0^{\hbar\Lambda} \left\{ \frac{\hbar\omega}{E_m + \hbar\omega - E - i\epsilon} \right. \\ &\quad \times \left[1 + \frac{1}{e^{\hbar\omega/kT} - 1} \right] + \frac{\hbar\omega}{E_m - \hbar\omega - E - i\epsilon} \\ &\quad \left. \times \left[\frac{1}{e^{\hbar\omega/kT} - 1} \right] \right\} \frac{R(\omega)}{\hbar} d(\hbar\omega). \end{aligned} \quad (38)$$

Ohmic dissipation

For the Ohmic dissipation $R(\omega) \equiv \eta = \text{const}$, in which we specialize from now on, we have the real part of the energy shift from Eq. (38) as

$$\begin{aligned} \text{Re} \Sigma_n(E_n) &= \frac{\eta}{\hbar\pi} \sum_m |\langle m|q|n \rangle|^2 \hbar\Lambda \\ &\quad - \frac{\eta}{\hbar\pi} \sum_m |\langle n|q|m \rangle|^2 (E_m - E_n) \\ &\quad \times \int_0^{\hbar\Lambda} \text{P} \left\{ \frac{1}{\hbar\omega + E_m - E_n} \right. \\ &\quad \left. + \left[\frac{1}{\hbar\omega + E_m - E_n} + \frac{1}{\hbar\omega - E_m + E_n} \right] \right. \\ &\quad \left. \times \frac{1}{e^{\hbar\omega/kT} - 1} \right\} d(\hbar\omega), \end{aligned} \quad (39)$$

where P stands for the principal value prescription. After subtracting the first term proportional to $\hbar\Lambda$ as a renormalization of the potential [13] following the prescription of Caldeira and Leggett [11], the real part $\text{Re} \Sigma_n(E_n)$ is rewritten as

$$\begin{aligned} \text{Re} \Sigma_n(E_n) &= -\frac{\eta}{\hbar\pi} \sum_m |\langle n|q|m \rangle|^2 (E_m - E_n) \\ &\quad \times \int_0^{\hbar\Lambda} \frac{1}{2} \left[\frac{1}{\hbar\omega + E_m - E_n} + \frac{1}{\hbar\omega - E_m + E_n} \right] \\ &\quad \times \left(1 + \frac{2}{e^{\hbar\omega/kT} - 1} \right) d(\hbar\omega) \\ &= -\frac{\eta}{\hbar\pi} \sum_m |\langle n|q|m \rangle|^2 (E_m - E_n) \\ &\quad \times \int_0^{\hbar\Lambda} \frac{1}{2} \text{P} \left[\frac{1}{\hbar\omega + E_m - E_n} + \frac{1}{\hbar\omega - E_m + E_n} \right] \\ &\quad \times \coth \left(\frac{\beta\hbar\omega}{2} \right) d(\hbar\omega) \end{aligned} \quad (40)$$

which agrees with the result of the field-theoretic formulation of the Caldeira-Leggett model [13]. It should be noted that the vacuum fluctuation term (spontaneous emission) in Eq. (1) plays a central role in our application, unlike the

conventional applications of the fluctuation-dissipation theorem where the vacuum fluctuation is usually subtracted [2].

For $\beta = 1/kT \rightarrow \infty$, one naturally recovers the zero temperature result [10]. For the *two-level approximation*, which is valid for the lowest two levels in a deep double-well potential, we have the result [13]

$$\text{Re } \Sigma_2(E_2) - \text{Re } \Sigma_1(E_1) \simeq \epsilon \bar{\eta} \ln(e^{-2} \beta \hbar \Lambda), \quad (41)$$

with the zeroth-order energy difference $\epsilon \equiv E_2 - E_1$ and $\bar{\eta} = (2\eta/\pi\hbar) | \langle 0|q|1 \rangle |^2$ for the temperature $\epsilon \ll 1/\beta \ll \hbar \Lambda$. One can confirm the absence of the $\epsilon \ln \epsilon$ dependence by splitting the integration range in Eq. (40) into $[0, \hbar \Lambda] = [0, a] + [a, \hbar \Lambda]$ with $\epsilon \ll a \ll 1/\beta \ll \hbar \Lambda$. The energy splitting (*order parameter* of quantum coherence) corrected by the dissipation is then given by

$$\begin{aligned} \epsilon_{(1)} &= [E_2 - \text{Re } \Sigma_2(E_2)] - [E_1 - \text{Re } \Sigma_1(E_1)] \\ &\simeq \epsilon [1 - \bar{\eta} \ln(e^{-2} \beta \hbar \Lambda)] \\ &\simeq \epsilon [(e^{-2}) \beta \hbar \Lambda]^{-\bar{\eta}} \end{aligned} \quad (42)$$

after the renormalization group improvement. This result, which suggests the suppression of quantum coherence $\epsilon_{(1)} \ll \epsilon$ for $\beta \hbar \Lambda \gg 1$, is in agreement with the dilute instanton analysis for the case of Ohmic dissipation [12]. Equation (42), when compared with the result $\epsilon_{(1)} = \epsilon [\hbar \omega_0 / \epsilon]^{-\bar{\eta}}$ or $\epsilon_{(1)} = \epsilon [\hbar \omega_0 / \epsilon]^{-\bar{\eta}/(1-\bar{\eta})}$ at $T=0$ [12,13], shows that the infrared cutoff, which was originally provided by ϵ , is replaced by $1/\beta$. We can thus analyze the quantum coherence without referring to an explicit model Lagrangian.

Although the mathematical basis of the dispersion relation at finite temperature is not as solid as the one at zero temperature, our relation (35) is justified in the present linear response approximation in a limited temperature region since it coincides with the second-order perturbation theory combined with Eqs. (15) and (16) [or Eqs. (30) and (31)]; we note that the diagonal matrix element $\langle n|q|n \rangle = 0$ for a specific double-well potential.

IV. FLUCTUATION-DISSIPATION THEOREM FOR FERMIONIC DISSIPATION

We now discuss a fermionic version of the fluctuation-dissipation theorem of Callen and Welton starting with Eq. (9). The operators Q and Q^\dagger are now taken to be fermionic operators and q and \bar{q} are Grassmann numbers that satisfy

$$q\bar{q} = -\bar{q}q, \quad q^2 = 0, \quad \bar{q}^2 = 0 \quad (43)$$

and

$$Qq = -qQ, \quad Q\bar{q} = -\bar{q}Q, \quad Q^2 = 0, \quad (Q^\dagger)^2 = 0 \quad (44)$$

Because of $Q^2 = 0$, the real fermionic case is trivial and we consider a complex Q that satisfies $QQ^\dagger \neq 0$. In a relativistic notation of four-dimensional space-time, our Q is regarded as one of the components of the (composite) two-component complex spinor Q_α , $\alpha = 1, 2$. (Instead, one may also con-

sider a set of real fermionic operators Q_1 and Q_2 with $Q_1^\dagger = Q_1$ and $Q_2^\dagger = Q_2$ corresponding to a Majorana spinor, which satisfy $Q_1 Q_2 \neq 0$.)

Equation (10) is now replaced by

$$\begin{aligned} P(\omega) &= \frac{\pi\omega}{2} q\bar{q} \int_0^\infty dE \rho(E) f(E) [| \langle E + \hbar\omega | Q^\dagger | E \rangle |^2 \\ &\quad \times \rho(E + \hbar\omega) + | \langle E - \hbar\omega | Q | E \rangle |^2 \rho(E - \hbar\omega)]. \end{aligned} \quad (45)$$

Note the relative sign of two terms in Eq. (45), which arises from the Grassmann nature of q and \bar{q} . We adopt the definition of the dissipative coefficient $P(\omega) = (\omega^2/2) R_f(\omega) q\bar{q}$ as in Eq. (13). We thus obtain

$$\begin{aligned} R_f(\omega) &= \frac{\pi}{\omega} \int_0^\infty dE \rho(E) f(E) [| \langle E + \hbar\omega | Q^\dagger | E \rangle |^2 \rho(E + \hbar\omega) \\ &\quad + | \langle E - \hbar\omega | Q | E \rangle |^2 \rho(E - \hbar\omega)] \\ &= \frac{\pi}{\omega} (1 + e^{-\beta\hbar\omega}) \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar\omega) \\ &\quad \times | \langle E + \hbar\omega | Q^\dagger | E \rangle |^2. \end{aligned} \quad (46)$$

From this expression of $R_f(\omega)$ we find the basic relations

$$\begin{aligned} \frac{2}{\pi} \frac{\hbar\omega}{2} \left[1 - \frac{1}{e^{\beta\hbar\omega} + 1} \right] R_f(\omega) \\ = \hbar \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar\omega) \\ \times | \langle E + \hbar\omega | Q^\dagger | E \rangle |^2, \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{2}{\pi} \frac{\hbar\omega}{2} \left[\frac{1}{e^{\beta\hbar\omega} + 1} \right] R_f(\omega) \\ = \hbar \int_0^\infty dE \rho(E) f(E) \rho(E - \hbar\omega) \\ \times | \langle E - \hbar\omega | Q | E \rangle |^2. \end{aligned} \quad (48)$$

The left-hand side of Eqs. (47) is regarded as the spontaneous and induced emissions of (effective) fermionic oscillators and the relative minus sign accounts for the Fermi statistics. The left-hand side of Eq. (48) is regarded as the (induced) absorption of the fermionic oscillators.

Equation (17) is now replaced by

$$\begin{aligned} \bar{R}_f(-\omega) &= \frac{\pi}{\omega} (1 + e^{-\beta\hbar\omega}) \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar\omega) \\ &\quad \times | \langle E + \hbar\omega | Q | E \rangle |^2, \end{aligned} \quad (49)$$

where we defined $P(-\omega) = (\omega^2/2) \bar{R}_f(\omega) \bar{q}q$ by changing the order of q and \bar{q} . For $\bar{R}_f(-\omega)$ with $\omega > 0$, we thus obtain the relations

$$\begin{aligned} & \frac{2}{\pi} \frac{\hbar \omega}{2} \left[1 - \frac{1}{e^{\beta \hbar \omega} + 1} \right] \bar{R}_f(-\omega) \\ &= \hbar \int_0^\infty dE \rho(E) f(E) \rho(E + \hbar \omega) \\ & \quad \times |\langle E + \hbar \omega | Q | E \rangle|^2, \end{aligned} \quad (50)$$

$$\begin{aligned} & \frac{2}{\pi} \frac{\hbar \omega}{2} \left[\frac{1}{e^{\beta \hbar \omega} + 1} \right] \bar{R}_f(-\omega) \\ &= \hbar \int_0^\infty dE \rho(E) f(E) \rho(E - \hbar \omega) \\ & \quad \times |\langle E - \hbar \omega | Q^\dagger | E \rangle|^2. \end{aligned} \quad (51)$$

Combining these relations, we finally obtain the fluctuation-dissipation theorem for fermionic dissipation (or fluctuation) as

$$\begin{aligned} \langle Q Q^\dagger \rangle & \equiv \int_0^\infty \langle E | Q Q^\dagger | E \rangle \rho(E) f(E) dE \\ &= \int_0^\infty dE \rho(E) f(E) \left\{ \int_0^\infty |\langle E + \hbar \omega | Q^\dagger | E \rangle|^2 \right. \\ & \quad \times \rho(E + \hbar \omega) d(\hbar \omega) + \int_0^\infty |\langle E - \hbar \omega | Q^\dagger | E \rangle|^2 \\ & \quad \left. \times \rho(E - \hbar \omega) d(\hbar \omega) \right\} \\ &= \frac{2}{\pi} \int_0^\infty d\omega \frac{\hbar \omega}{2} \left\{ \left[1 - \frac{1}{e^{\beta \hbar \omega} + 1} \right] R_f(\omega) \right. \\ & \quad \left. + \left[\frac{1}{e^{\beta \hbar \omega} + 1} \right] \bar{R}_f(-\omega) \right\} \end{aligned} \quad (52)$$

and similarly

$$\begin{aligned} \langle Q^\dagger Q \rangle &= \frac{2}{\pi} \int_0^\infty d\omega \frac{\hbar \omega}{2} \left\{ \left[\frac{1}{e^{\beta \hbar \omega} + 1} \right] R_f(\omega) \right. \\ & \quad \left. + \left[1 - \frac{1}{e^{\beta \hbar \omega} + 1} \right] \bar{R}_f(-\omega) \right\}. \end{aligned} \quad (53)$$

Furthermore,

$$\frac{1}{2} \langle Q^\dagger Q - Q Q^\dagger \rangle = \frac{2}{\pi} \int_0^\infty d\omega E_f(\omega, T) \left[\frac{R_f(\omega) - \bar{R}_f(-\omega)}{2} \right], \quad (54)$$

where

$$E_f(\omega, T) = -\frac{\hbar \omega}{2} + \frac{\hbar \omega}{e^{\beta \hbar \omega} + 1}. \quad (55)$$

Our sign convention of the dissipative coefficients $R_f(\omega)$ and $\bar{R}_f(-\omega)$ is chosen so that $R_f(\omega) \geq 0$ and $\bar{R}_f(-\omega) \geq 0$.

In practice, the composite operator Q may carry a well-defined fermion number and the fermion number may be conserved. Moreover, the production of an antiparticle (or hole state) may be suppressed; in such a case, one may set $\bar{R}_f(-\omega) = 0$ in our formulas. The physical content of the fluctuation-dissipation theorem for fermionic dissipation, as is formulated here, is that the thermal average of the (composite) operator $Q Q^\dagger$ or $Q^\dagger Q$, which characterizes the fluctuation, is represented in terms of effective fermionic excitations with their spectrum being specified by $R_f(\omega)$; the parameter $R_f(\omega)$ in turn characterizes the energy dissipation into the dissipative medium. In this context, the presence of energy dissipation with $R_f(\omega)$ necessarily leads to the presence of effective (or collective) fermionic excitations; this property is analogous to the Nambu-Goldstone theorem for spontaneous symmetry breakdown, which asserts an inevitable appearance of massless excitations when continuous symmetry is spontaneously broken, as was emphasized in Ref. [10]. The difference is that these collective excitations in the present context are *effective* ones.

We finally comment on a technical complication in deriving Eq. (46). The calculation (46) may appear to be straightforward and identical to Eq. (14). However, a closer examination reveals that an additional assumption is in fact involved: If $|E\rangle$ is a bosonic state (with even “ G parity” [15]), i.e.,

$$q|E\rangle = |E\rangle q, \quad (56)$$

then $Q|E\rangle$ is a fermionic state (odd G parity), i.e.,

$$q(Q|E\rangle) = -(Q|E\rangle)q \quad (57)$$

and vice versa. Since $\langle E \pm \hbar \omega | Q | E \rangle$ is not a Grassmann variable but rather an ordinary number, $|E \pm \hbar \omega\rangle$ should have different G parity from $|E\rangle$. That is, the interaction $q e^{i\omega t} Q + Q^\dagger \bar{q} e^{-i\omega t}$ would induce not only the energy shift $E \rightarrow E \pm \hbar \omega$ but also the change of G parity. This fact prevents one from obtaining Eq. (46) in a naive way because the state $|E\rangle$ obtained from $|E - \hbar \omega\rangle$ by a shift in E , $E \rightarrow E + \hbar \omega$, has a different G parity from the original $|E\rangle$. In obtaining Eq. (46) we assume that the initial state $|E\rangle$ has even G parity with probability 1/2 and odd G parity with probability 1/2. This assumption is consistent with the notion of dissipation, which implies that the “radiation” carries away or injects a small amount of energy specified by $\hbar \omega$ at a time. We can thus obtain Eq. (46) by averaging over G parity also.

V. CONCLUSION

The main purpose of the present paper is to point out the remarkable relations (1) and (2), which arise from a simple application of Fermi’s golden rule and the weak temperature dependence of the dissipative coefficient, and their applications to quantum tunneling with dissipation. In the conventional treatment of the fluctuation-dissipation theorem in Eq. (25), the vacuum fluctuation is often subtracted away by simply saying that it is not observable [14]. In contrast, the term corresponding to the vacuum fluctuation plays a central role in our application since it describes the spontaneous emission of effective excitations into the dissipative medium; it is

thus the only effect remaining at the vanishing temperature.

Our dispersion relation for the self-energy (35), which is essentially equivalent to the second-order perturbation theory in linear response approximation, then gives rise to a change in the real part of the tunneling energy splitting (which is the order parameter of quantum coherence). Our formulas naturally give rise to the dilute instanton results for the order parameter both at finite and vanishing temperature [12]. In this sense our attempt to reproduce the physical results of the Caldeira-Leggett model without referring to an explicit model Lagrangian has been successful at least in the analyses of quantum coherence at both $T=0$ [10] and $T\neq 0$.

Of course, the formulation of Caldeira and Leggett [11] is flexible enough and it is applicable to many other physical contexts. Nevertheless, for those who wonder if one can analyze some physical processes without an infinite number of oscillators, our reformulation of the quantum tunneling with dissipation on the basis of the fluctuation-dissipation theorem and dispersion relations may give an answer by showing such possibility as well as limitations.

Another purpose of the present paper is to present a generalization of the fluctuation-dissipation theorem of Callen-Welton to the case of fermionic dissipation (or fluctuation). Since fermions are basically quantum mechanical, the notion of fermionic dissipation is characteristically quantum mechanical. We emphasize that a mere excitation of a fermion from one state to another does not imply fermionic fluctuation in the present context; the force fields or currents involved should be fermionic, though the fermionic excitation may be an effective one or a quasiparticle. In full quantum theory, both the bosonic and fermionic modes can be equally

added to or removed from the system we are interested in and a fully quantum mechanical fluctuation-dissipation theorem should be able to handle the fermionic fluctuation and, consequently, fermionic dissipation.

In the context of condensed matter physics, we usually measure *bosonic* quantities such as voltage or electric current even if the elementary process involves the transfer of fermions. In such a case, the conventional bosonic fluctuation-dissipation theorem is applicable. It is our hope that a properly defined treatment of elementary transfer processes may lead to an application of the notion of fermionic fluctuation or dissipation in the future.

As for the physical phenomena where the fermionic fluctuation-dissipation theorem may have some relevance, we note the fermion emission from a black hole [16] and the fermion production in an accelerated frame [17]. The notion of quantum noise plays a fundamental role in the analyses of these processes and thus the fermionic fluctuation-dissipation theorem may provide a convenient framework to describe some general features of these interesting quantum processes. Another area of physics where fermionic fluctuation may play a role is the theory of supersymmetry (or boson-fermion symmetry) [18]. The basic current of supersymmetry is *fermionic* and thus the thermal average of a product of such currents (or related gravitino fields) inevitably leads to a notion of fermionic fluctuation. Though we do not know a specific application of the fermionic fluctuation-dissipation theorem at this moment, our formulation may turn out to be useful in future analyses of supersymmetry in a multiparticle thermal setting, where supersymmetry is known to be inevitably broken by thermal effects.

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